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Mathematical modeling of three equal collinear cracks in an orthotropic solid

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Abstract We consider a homogeneous elastic, orthotropic solid containing three equal collinear cracks, loaded in tension by symmetrically distributed normal stresses. Following Guz's representation theorem and solving Riemann-Hilbert problems we determine the expressions of the complex potentials. Using the asymptotic analysis, we obtain the asymptotic values of the incremental stress and displacement fields. We determine the tangential stresses near the crack tips. Using the maximum tangential stress criterion and numerical computations we study the interaction problem for a Graphite-epoxy fiber reinforced composite material.

Keywords Three equal collinear cracks · Riemann-Hilbert problem · maximum tangential stress criterion · cracks interaction.

1 Introduction

Crack initiation, propagation direction, crack tip fields and cracks interaction in static orthotropic plane linear elasticity are the main themes for mathematical modelling and simulation and represent important problems of Fracture mechanics. Assume that the admissible equilibrium states of the body are

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plane strain states relative to Ox_1x_2 plane. In this case the equilibrium states of the material can be represented by two complex potentials defined in two complex planes [1-10]. We shall use Guz's representation of the elastic state [1], without initial deformation, in a weakly modified form due to Soos [10-17].

Our first aim is to determine the elastic state produced in the body using complex potentials. Numerous studies regarding this subject were done in the literature [18-23]. Analytic solutions for stress distribution and crack interaction for a composite material containing two collinear cracks can be found in the works [24-29]. The problem of three collinear crack interaction, elastic fields distribution and crack propagation using the integral equations method were studied in [30-34]. Mukherjee and Das [31] considered the problem for three interfacial cracks between two dissimilar orthotropic media. Using the theories of stress intensity factors and strain energy release rate they studied the interaction cracks for a pair of composite materials.

We suppose that our material is unbounded and contains three equal collinear cracks situated in the same plane Ox_1x_2 (see Fig. 1). We formulate and give the solution of the mathematical problem in Section 2, assuming that the applied normal stresses have a given constant value.

Our second aim is to find which tip of the cracks will start to propagate first. To do this we determine in the Section 3 singular parts of the elastic states near the crack tips using the asymptotic method and following the approach used in [35], for three equal, collinear cracks in Mode II of fracture.

Several fracture criteria have been suggested in the literature. Our third aim is to extend the maximum tangential stress $\sigma_{\theta\theta}$ - criterion (MTS) due to Erdogan and Sih [36-37] and to find which tip will propagate first. For doing this we consider as an example two configurations of three equal collinear cracks in a Graphite-epoxy fiber reinforced elastic composite. Using numerical computations, we determine that:
- in the case when the distance between the cracks is much smaller than their length, *i.e.* there exist a strong interaction between the cracks, the inner tips of the cracks start to propagate first and cracks tend to unify;
- in the case when the distance between the cracks is much greater than their length, *i.e.* there exist a weak interaction between the cracks and all cracks start to propagate almost in the same time.

2 Mathematical problem

We consider an orthotropic, linear elastic material, representing a fiber reinforced composite. We assume that the equilibrium states of the material are plane strain states relative to Ox_1x_2 plane.

As it was shown by Guz ([1], [2]) in the assumed circumstances the elastic state of the body can be expressed by two analytic complex potentials $\Phi_j(z_j)$ defined in two complex planes z_j, $j = 1, 2$.

We present Guz's representation formulae for the case of non-equal roots.

We have:

$$u_1 = 2\,\mathrm{Re}\{b_1\Phi_1(z_1) + b_2\Phi_2(z_2)\}, \qquad u_2 = 2\,\mathrm{Re}\{c_1\Phi_1(z_1) + c_2\Phi_2(z_2)\}$$
$$\sigma_{11} = 2\,\mathrm{Re}\{\mu_1^2\Psi_1(z_1) + \mu_2^2\Psi_2(z_2)\}, \qquad \sigma_{22} = 2\,\mathrm{Re}\{\Psi_1(z_1) + \Psi_2(z_2)\}$$
$$\sigma_{12} = \sigma_{21} = -2\,\mathrm{Re}\{\mu_1\Psi_1(z_1) + \mu_2\Psi_2(z_2)\} \tag{2.1}$$

where $u_j, j = \overline{1,2}$ represent the component of the displacement field and σ_{ij}, $i, j = \overline{1,2}$ represent the components of the nominal stress.

In these relations, we denoted by

$$\Psi_j(z_j) = \Phi_j'(z_j) = \frac{d\Phi_j(z_j)}{dz_j},\ j = 1,2 \tag{2.2}$$

and

$$z_j = x_1 + \mu_j x_2,\ j = 1,2. \tag{2.3}$$

The parameters μ_j are the roots of the algebraic equation

$$\mu^4 + 2A\mu^2 + B = 0 \tag{2.4}$$

with

$$A = \frac{\omega_{1111}\omega_{2222} + \omega_{1221}\omega_{2112} - (\omega_{1122} + \omega_{1212})^2}{2\omega_{2222}\omega_{2112}}, \quad B = \frac{\omega_{1111}\omega_{1221}}{\omega_{2112}\omega_{2222}}. \tag{2.5}$$

The parameters $b_j, c_j, (j = 1,2)$ have the following relations:

$$b_j = -(\omega_{1122} + \omega_{1212})/B_j, \quad c_j = (\omega_{2112}\mu_j^2 + \omega_{1111})/(B_j\mu_j) \tag{2.6}$$

with

$$B_j = \omega_{2222}\omega_{2112}\mu_j^2 + \omega_{1111}\omega_{2222} - \omega_{1122}(\omega_{1122} + \omega_{1212}).$$

The compliance coefficients ω_{klmn} $(k, l, m, n = 1, 2)$ can be expressed through engineering constants of the composite by the following relations:

$$\omega_{1111} = \frac{1 - \nu_{23}\nu_{32}}{E_2E_3H}, \ \omega_{2222} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3H}, \ \omega_{1122} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1E_3H}, \tag{2.7}$$
$$\omega_{1212} = \omega_{1221} = \omega_{2112} = G_{12},$$

with

$$H = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - \nu_{21}\nu_{32}\nu_{13} - \nu_{12}\nu_{23}\nu_{31}}{E_1E_2E_3}. \tag{2.8}$$

In these relations E_1, E_2, E_3 are Young's moduli in the corresponding symmetry directions of the material, G_{12} is the shear modulus in the symmetry plane Ox_1x_2 and $\nu_{12}, ..., \nu_{32}$ are the Poisson's ratios.

Also, for an orthotropic material the roots μ_1 and μ_2 usually are not equal. In what follows we consider this case of non-equal roots,

$$\mu_1 \neq \mu_2. \tag{2.9}$$

Let us assume that we have an unbounded composite which contains three equal collinear cracks situated in the same plane Ox_1x_3, the Ox_2 axis being perpendicular to the cracks faces. We consider that our cracked composite is loaded by remote normal stress p, (see Fig. 1). So, we can use Guz's complex representation to study our problem corresponding to the first mode in classical fracture mechanics.

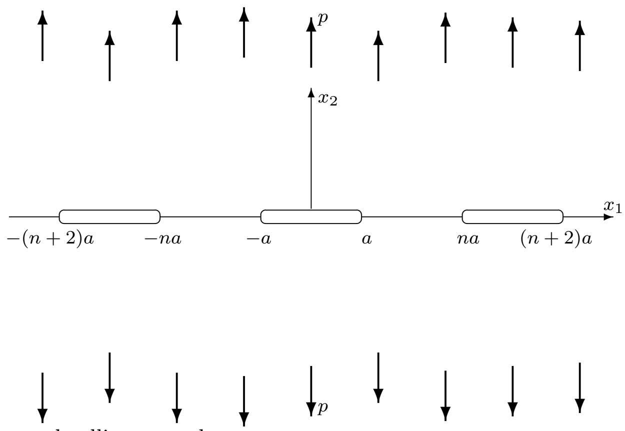

Fig. 1 Three equal collinear cracks

We denote by $\mathscr{C}$ the cut corresponding to the segments $(-(n+2)a, -na)$, $(-a, a)$ and $(na, (n+2)a)$. Here $a > 0$ is a given positive constant and n is a real number greater than 1.

According to the considered mechanical problem and by means of the principle of superposition, the cracks faces are loaded in tension by normal stresses $p = p(t)$, $t \in \mathscr{C}$, symmetrically distributed relative to the plane containing the cracks. The components σ_{21} and σ_{22} of the nominal stress must satisfy the following boundary conditions:

$$\sigma_{21}(t, 0^+) = \sigma_{21}(t, 0^-) = 0, \qquad \sigma_{22}(t, 0^+) = \sigma_{22}(t, 0^-) = -p(t), \tag{2.10}$$

for $t \in \mathscr{C}$.

Also, at large distances from the cracks the displacements and stresses must vanish; *i.e.*

$$\lim_{r \to \infty} \{u_i(x_1, x_2), \sigma_{ij}(x_1, x_2)\} = 0, \ r = \sqrt{x_1^2 + x_2^2}, \ i, \ j = 1, 2. \tag{2.11}$$

From (**??**) we conclude that the complex potentials have to fulfill the following far field restrictions

$$\lim_{z_j \to \infty} \{\Phi_j(z_j), \Psi_j(z_j)\} = 0, \ j = 1, 2. \tag{2.12}$$

We denote by $f^+(t)$ and $f^-(t)$ the upper and lower limits of a complex valued function $f(z)$, $z = x_1 + ix_2$, *i.e.*

$$f^+(t) = \lim_{\substack{z \to t, \\ x_2 > 0}} f(z) \quad \text{and} \quad f^-(t) = \lim_{\substack{z \to t, \\ x_2 < 0}} f(z). \tag{2.13}$$

From the Guz's representation formula $(\mathbf{??})_5$ and from the boundary condition $(\mathbf{??})_1$ we get:

$$\begin{aligned} &\mu_1\Psi_1^+ + \mu_2\Psi_2^+ + \overline{\mu_1}\bar{\Psi}_1^- + \overline{\mu_2}\bar{\Psi}_2^- = 0,\\ &\mu_1\Psi_1^- + \mu_2\Psi_2^- + \overline{\mu_1}\bar{\Psi}_1^+ + \overline{\mu_2}\bar{\Psi}_2^+ = 0 \end{aligned} \tag{2.14}$$

where the superposed bar denotes the complex conjugation.

Adding and subtracting the above relations, using the well known proprieties of Cauchy's integral ([4], Ch. 4-6) and following the same procedure used by Craciun and Soos [3] we obtain that the complex potentials involved in our homogeneous Riemann-Hilbert problems have to satisfy in the whole complex plane the restriction

$$\Psi_2(z) = -\frac{\mu_1}{\mu_2}\Psi_1(z). \tag{2.15}$$

Now, using $(\mathbf{??})$ in the second boundary condition $(\mathbf{??})_2$, we obtain that the complex potential $\Psi_1(z_1)$ must satisfy the following equations:

$$\rho\Psi_1^+(z) + \bar{\rho}\bar{\Psi}_1^-(z) = -p(t), \quad \rho\Psi_1^-(z) + \bar{\rho}\bar{\Psi}_1^+(z) = -p(t) \tag{2.16}$$

for $t \in \mathscr{C}$, where we denoted

$$\rho = \frac{\Delta}{\mu_2}, \tag{2.17}$$

and

$$\Delta = \mu_2 - \mu_1. \tag{2.18}$$

Adding and subtracting the relations $(\mathbf{??})$, we get the following equivalent restrictions:

$$\big(\rho\Psi_1(t) + \bar{\rho}\bar{\Psi}_1(t)\big)^+ + \big(\rho\Psi_1(t) + \bar{\rho}\bar{\Psi}_1(t)\big)^- = -2p(t), \tag{2.19}$$

$$\big(\rho\Psi_1(t) - \bar{\rho}\bar{\Psi}_1(t)\big)^+ - \big(\rho\Psi_1(t) - \bar{\rho}\bar{\Psi}_1(t)\big)^- = 0$$

for $t \in \mathscr{C}$.

From $(\mathbf{??})_2$ we obtain the following relation

$$\rho\Psi_1(z) - \bar{\rho}\bar{\Psi}_1(z) = 0. \tag{2.20}$$

The restriction $(\mathbf{??})_1$ is equivalent with a nonhomogeneous Riemann-Hilbert problem. Following Muskhelishvili's formalism ([**?**], Ch. 6), we get the following expression of the complex potentials:

$$\begin{aligned} \Psi_1(z_1) &= -\frac{\mu_2}{\Delta}\frac{X(z_1)}{2\pi i}\int_{\mathscr{C}}\frac{p(t)dt}{X^+(t)(t-z_1)} + \frac{\mu_2}{2\Delta}P(z_1)X(z_1)\\ \Psi_2(z_2) &= \frac{\mu_1}{\Delta}\frac{X(z_2)}{2\pi i}\int_{\mathscr{C}}\frac{p(t)dt}{X^+(t)(t-z_2)} - \frac{\mu_1}{2\Delta}P(z_2)X(z_2), \end{aligned} \tag{2.21}$$

where $P(z)$ is an arbitrary polynomial with complex coefficients and we denoted by $\chi(z)$ Plemelj's function *i.e.*

$$X(z) = \frac{1}{\chi(z)} \tag{2.22}$$

with

$$\chi(z) = \sqrt{(z^2 - a^2)(z^2 - n^2a^2)[z^2 - (n+2)^2a^2]}. \tag{2.23}$$

Taking into account the far field restriction (**??**) the polynomial $P(z)$ has second degree .

Let us assume now that the crack faces are subjected to a given stress having a constant value p, *i.e.*

$$p(t) = p = const. > 0, \quad \text{for} \quad t \in \mathscr{C}. \tag{2.24}$$

Using the theory of complex functions ([4], Chap. 6) the integrals involved in (**??**) take the following form:

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \frac{p dt}{X^+(t)(t - z_k)} = \frac{p}{2}\left[\frac{1}{X(z_k)} - (\alpha_3 z_k^3 + \alpha_2 z_k^2 + \alpha_1 z_k + \alpha_0)\right] \tag{2.25}$$

for $k = 1, 2$.

So, using (**??**) in (**??**) we obtain the following expressions for our complex potentials:

$$\begin{aligned} \Psi_1(z_1) = \frac{\mu_2}{2\Delta} p \left\{ \frac{\alpha_3 z_1^3 + \alpha_2 z_1^2 + \alpha_1 z_1 + \alpha_0}{\sqrt{(z_1^2 - a^2)(z_1^2 - n^2a^2)[z_1^2 - (n+2)^2a^2]}} - 1 \right\} + \\ \frac{\mu_2}{2\Delta} \frac{P(z_1)}{\sqrt{(z_1^2 - a^2)(z_1^2 - n^2a^2)[z_1^2 - (n+2)^2a^2]}}, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \Psi_2(z_2) = -\frac{\mu_1}{2\Delta} p \left\{ \frac{\alpha_3 z_2^3 + \alpha_2 z_2^2 + \alpha_1 z_1 + \alpha_0}{\sqrt{(z_2^2 - a^2)(z_2^2 - n^2a^2)[z_2^2 - (n+2)^2a^2]}} - 1 \right\} - \\ \frac{\mu_1}{2\Delta} \frac{P(z_2)}{\sqrt{(z_2^2 - a^2)(z_2^2 - n^2a^2)[z_2^2 - (n+2)^2a^2]}} \end{aligned} \tag{2.27}$$

with

$$\alpha_3 = 1, \quad \alpha_2 = 0, \quad \alpha_1 = -\frac{2n^2 + 4n + 5}{2} a^2, \quad \alpha_0 = 0. \tag{2.28}$$

According to (**??**) we have

$$\Phi(z_j) = \int \Psi(z_j) dz_j, j = 1, 2. \tag{2.29}$$

To assure the uniformity of displacement field u expressed through complex potentials $\Phi_j(z_j)$ as in $(\textbf{??})_{1,2}$, we have to guarantee the uniformity of the potentials $\Phi_j(z_j)$ on closed path around the three cracks.

This requirement uniquely determines the polynomials $P(z_j)$ (see the Appendix A) and leads to the following final expressions of the complex potentials $\Psi_j(z_j)$

$$\Psi_1(z_1) = \frac{p\mu_2}{2\Delta}\left(\frac{z_1^3 - \frac{J_3}{J_1}z_1}{\chi(z_1)} - 1\right), \quad \Psi_2(z_2) = -\frac{p\mu_1}{2\Delta}\left(\frac{z_2^3 - \frac{J_3}{J_1}z_2}{\chi(z_2)} - 1\right) \tag{2.30}$$

where

$$J_k = \int_{na}^{(n+2)a} \frac{t^k}{A(t)}dt, \quad \text{for} \quad k = 1, 2, 3 \tag{2.31}$$

and $A(t)$ is defined by:

$$A(t) = \sqrt{(a^2 - t^2)(n^2a^2 - t^2)((n+2)^2a^2 - t^2)}. \tag{2.32}$$

3 Asymptotic crack tip values

Due to the symmetry of the configuration of our orthotropic linear elastic composite, we start to determine the asymptotic values of the complex potentials $\Psi(z_j)$ and of the stress field σ_{ij}, $i, j = 1, 2$ for the crack tips a, na and $(n+2)a$.

To obtain these values, we have to know the nonregular parts as asymptotic values of the potentials in a small neighborhood of the crack tip. We denote by x_1 and x_2 the Cartesian coordinates of a current point situated in the vicinity of the crack tip a. We have

$$x_1 = a + r\cos\theta \ , \quad x_2 = r\sin\theta \ , \tag{3.1}$$

where the polar coordinates r and θ designate, respectively, the radial distance from the crack tip and the angle between the Ox_1 axis and the radial line in trigonometric sense. Using (**??**) and (**??**) we get

$$z_j - a = r(\cos\theta + \mu_j \sin\theta), \quad j = 1, 2. \tag{3.2}$$

Introducing the functions

$$\Gamma_j(\theta) = \cos\theta + \mu_j \sin\theta \ , \quad j = 1, 2 \tag{3.3}$$

the above relations (**??**) become

$$z_j - a = r\Gamma_j(\theta), \quad j = 1, 2. \tag{3.4}$$

In a small vicinity of the crack tip a we have

$$z_j \approx a \ , \ j = 1, 2 \tag{3.5}$$

and the following approximate equation:

$$\chi(z_j) \approx \sqrt{2ar}\, a^2 \sqrt{\Gamma_j(\theta)(n+1)^2(n-1)(n+3)} \ , \ j = 1, 2. \tag{3.6}$$

From (??) and (??) we obtain the asymptotic on nonregular parts of the complex potentials in a small neighborhood of the tip a

$$\Psi_1(z_1) \approx \frac{p}{2\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_2}{\sqrt{\Gamma_1(\theta)}}m_1, \quad \Psi_2(z_2) \approx -\frac{p}{2\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_1}{\sqrt{\Gamma_2(\theta)}}m_1 \tag{3.7}$$

where

$$m_1 = m_1(n) = \frac{1 - \frac{J_3}{J_1 a^2}}{(n+1)\sqrt{(n-1)(n+3)}}\ . \tag{3.8}$$

The asymptotic value of the stresses σ_{ij}, $i,j = 1,2$ near the tip are obtained using the representations $(??)_{3-6}$ and the relations (??) :

$$\begin{aligned}
\sigma_{11}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_1 \operatorname{Re}\left\{\frac{\mu_1\mu_2}{\Delta}\left(\frac{\mu_1}{\sqrt{\Gamma_1(\theta)}} - \frac{\mu_2}{\sqrt{\Gamma_2(\theta)}}\right)\right\},\\
\sigma_{12}(r,\theta) = \sigma_{21}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_1 \operatorname{Re}\left\{\frac{\mu_1\mu_2}{\Delta}\left(\frac{-1}{\sqrt{\Gamma_1(\theta)}} + \frac{1}{\sqrt{\Gamma_2(\theta)}}\right)\right\},\\
\sigma_{22}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_1 \operatorname{Re}\left\{\frac{1}{\Delta}\left(\frac{\mu_2}{\sqrt{\Gamma_1(\theta)}} - \frac{\mu_1}{\sqrt{\Gamma_2(\theta)}}\right)\right\}.
\end{aligned} \tag{3.9}$$

We consider now the crack tip na.

In a similar manner as for the crack tip a, it leads us to the following asymptotic values:

$$\Psi_1(z_1) \approx \frac{p}{2i\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_2}{\sqrt{\Gamma_1(\theta)}}m_2, \quad \Psi_2(z_2) \approx -\frac{p}{2i\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_1}{\sqrt{\Gamma_2(\theta)}}m_2 \tag{3.10}$$

where

$$m_2 = m_2(n) = \frac{n^3 - n\frac{J_3}{J_1 a^2}}{2(n+1)\sqrt{(n-1)n}} \tag{3.11}$$

and

$$\begin{aligned}
\sigma_{11}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_2 \operatorname{Re}\left\{\frac{\mu_1\mu_2}{i\Delta}\left(\frac{\mu_1}{\sqrt{\Gamma_1(\theta)}} - \frac{\mu_2}{\sqrt{\Gamma_2(\theta)}}\right)\right\},\\
\sigma_{12}(r,\theta) = \sigma_{21}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_2 \operatorname{Re}\left\{\frac{\mu_1\mu_2}{i\Delta}\left(\frac{-1}{\sqrt{\Gamma_1(\theta)}} + \frac{1}{\sqrt{\Gamma_2(\theta)}}\right)\right\},\\
\sigma_{22}(r,\theta) &\approx \frac{p\sqrt{a}}{\sqrt{2r}}m_2 \operatorname{Re}\left\{\frac{1}{i\Delta}\left(\frac{\mu_2}{\sqrt{\Gamma_1(\theta)}} - \frac{\mu_1}{\sqrt{\Gamma_2(\theta)}}\right)\right\}.
\end{aligned} \tag{3.12}$$

In these relations r is the radial distance from the crack tip na to an arbitrary point in the vicinity of considered tip.

To get the asymptotic values near the crack tip $(n+2)a$, we use the same procedure and we obtain

$$\Psi_1(z_1) \approx \frac{p}{2\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_2}{\sqrt{\Gamma_1(\theta)}}m_3, \quad \Psi_2(z_2) \approx -\frac{p}{2\Delta}\sqrt{\frac{a}{2r}}\frac{\mu_1}{\sqrt{\Gamma_2(\theta)}}m_3 \tag{3.13}$$

where

$$m_3 = m_3(n) = \frac{(n+2)^3 - (n+2)\frac{J_3}{J_1 a^2}}{2(n+1)\sqrt{(n+2)(n+3)}} \tag{3.14}$$

and the asymptotic value of the normal stresses σ_{ij}, $i, j = 1, 2$ have the same representation as in (**??**) but we have to replace m_1 from (**??**) with m_3 from (**??**).

4 Numerical example using tangential stress criterion

In this Section we extend the maximum tangential stress criterion from isotropic brittle materials ([36]-[37]) to orthotropic materials. We intend to observe which tip of the cracks will propagate first for a particular orthotrop composite material.

Erdogan and Sih's maximum tangential stress criterion states the following hypothesis for the extension of cracks in a brittle material under slowly applied plane loads:

- The crack extension starts at its tip in radial direction.
- The crack extension starts in the plane perpendicular to the direction of greatest tension.

These hypotheses imply that the crack will start to grow from the tip in the direction along which the tangential stress $\sigma_{\theta\theta}$ is maximum and the shear stress $\sigma_{r\theta}$ is zero, (*i.e.* $\sigma_{\theta\theta}$ is a principal stress and the shear stress $\sigma_{r\theta}$ vanishes for that direction).

Mathematically, the above hypothesis are expressed by

$$\sigma_{\theta\theta}(\theta_c) = \sigma_c, \quad \frac{\partial \sigma_{\theta\theta}}{\partial \theta}(\theta_c) = 0, \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta}(\theta_c) < 0. \tag{4.1}$$

Let us consider a Graphite-epoxy fiber - reinforced orthotropic composite material characterized by the following engineering constants ([2])

$$\begin{gathered} E_1 = 190GPa, \;\; E_2 = E_3 = 10GPa, \;\; G_{12} = 7GPa, \\ G_{13} = G_{23} = 6GPa, \;\; \nu_{12} = 0.3, \;\; \nu_{13} = \nu_{23} = 0.2, \end{gathered} \tag{4.2}$$

where E_1, E_2, E_3 are Young's moduli in the corresponding symmetry directions of the material, ν_{12}, ..., ν_{23} are the Poisson's ratios and G_{12}, G_{13}, G_{23} are the shear moduli.

The compliance coefficients ω_{klmn} $(k,l,m,n = 1,2)$ involved in the relations (??)-(??) for our composite material (??) have the following values:

$$\omega_{1111} = 191.62GPa, \quad \omega_{1212} = \omega_{1221} = \omega_{2112} = 7GPa,$$
$$\omega_{2222} = 10.48GPa, \quad \omega_{1122} = 3.57GPa. \tag{4.3}$$

With these values in (??) we get the parameters A and B

$$A = 13.26; \quad B = 18.28 \tag{4.4}$$

and our algebraic equation (??) become:

$$\mu^4 + 26.52\mu^2 + 18.28 = 0, \tag{4.5}$$

with the roots $\mu_1 = \bar{\mu_3}$ and $\mu_2 = \bar{\mu_4}$:

$$\mu_1 = 5,08i, \quad \mu_2 = 0,84i. \tag{4.6}$$

We observe that in this case the roots of eq. (??) are note equal as supposed before.

The physical components of the stress, the tangential stress $\sigma_{\theta\theta}$ and the shear stress $\sigma_{r\theta}$ are given by:

$$\sigma_{\theta\theta} = \sigma_{11}\sin^2\theta - 2\sigma_{12}\sin\theta\cos\theta + \sigma_{22}\cos^2\theta,$$
$$\sigma_{r\theta} = \sigma_{12}\cos 2\theta + \frac{\sigma_{22}-\sigma_{11}}{2}\sin 2\theta. \tag{4.7}$$

To determine the propagation of crack tips we find the maximum value of the tangential stress $\sigma_{\theta\theta}$, for all crack tips. The crack will start to propagate in a perpendicular direction to the direction of θ_c from the tip for which $\sigma_{\theta\theta}$ is maximum.

Due to the symmetry of the problem we shall study only the behavior of the tips a, na, $(n+2)a$.

In the Figs. 2 - 4, using the numerical computations, we have represented the normalized tangential stress $\sigma_{\theta\theta}$ and normalized shear stress $\sigma_{r\theta}$ corresponding to above mentioned tips a, na, $(n+2)a$, versus θ and n. The stresses were normalized by parameter α given by the relation:

$$\alpha = \frac{1}{p}\sqrt{\frac{2r}{a}}. \tag{4.8}$$

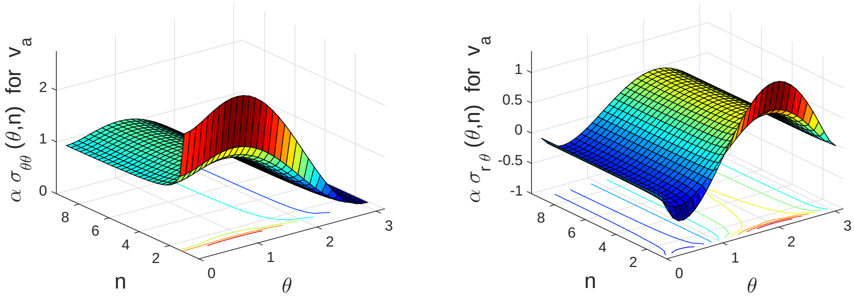

Fig. 2 $\sigma_{\theta\theta}$ (left) and $\sigma_{r\theta}$ (right) normalized with $\alpha = \frac{\sqrt{2r}}{p\sqrt{a}}$ for crack tip a

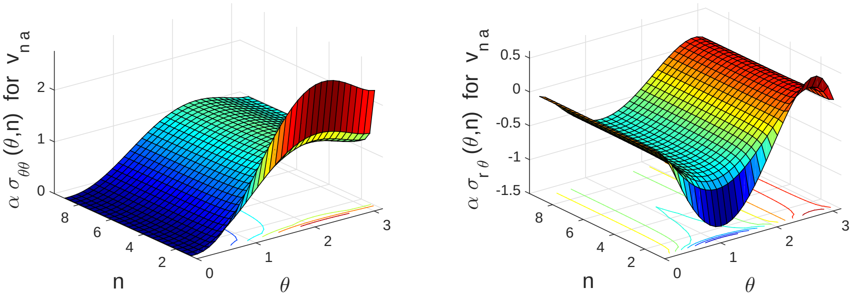

Fig. 3 $\sigma_{\theta\theta}$ (left) and $\sigma_{r\theta}$ (right) normalized with $\alpha = \frac{\sqrt{2r}}{p\sqrt{a}}$ for crack tip na

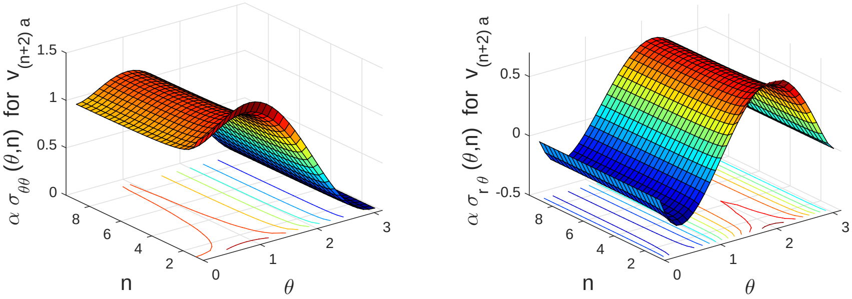

Fig. 4 $\sigma_{\theta\theta}$ (left) and $\sigma_{r\theta}$ (right) normalized with $\alpha = \frac{\sqrt{2r}}{p\sqrt{a}}$ for crack tip $(n+2)a$

Let us denote by $\sigma_{\theta\theta}^{(v)}$ and $\sigma_{r\theta}^{(v)}$ the tangential and shear stress, respectively, normalized by α (**??**) and corresponding to the tip $v \in \{a, na, (n+2)a\}$, and by $\sum_{\theta\theta}^{(v)}$ - the maximum value of the normalized tangential stress corresponding to the tip v, *i.e.*

$$\varSigma_{\theta\theta}^{(v)} = \max_{\theta\in[0,\pi]} \sigma_{\theta\theta}^{(v)}, \tag{4.9}$$

for $v \in \{a, na, (n+2)a\}$.

We consider first case $n = 2$ (interacting cracks).

Using the eqs (**??**), (**??**), (**??**)-(**??**), (**??**)-(**??**) and mathematical software we ploted the normalized tangential stress $\sigma_{\theta\theta}^{(v)}$ in Fig. 5 (left) and respectively the normalized shear stress $\varSigma_{r\theta}^{(v)}$ in Fig. 5 (right) corresponding to $v \in \{a, na, (n+2)a\}$ and parameter α (**??**).

We observe, see Fig. 5 (left), that the values $\varSigma_{\theta\theta}^{(v)}$ satisfy the inequality:

$$\varSigma_{\theta\theta}^{((n+2)a)} < \varSigma_{\theta\theta}^{(na)} \leq \varSigma_{\theta\theta}^{(a)} \tag{4.10}$$

and the values $\varSigma_{\theta\theta}^{(a)}$ and $\varSigma_{\theta\theta}^{(na)}$ differ by a small quantity.

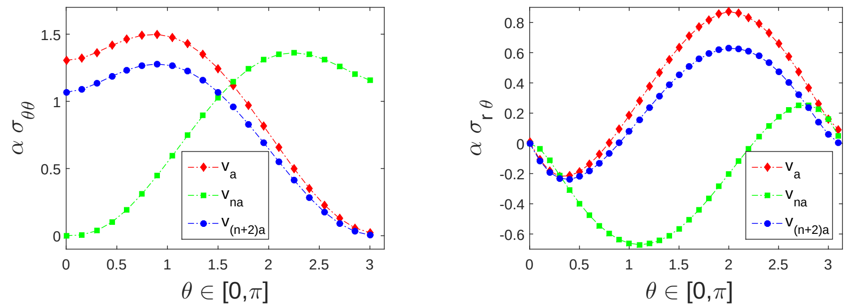

Fig. 5 $\sigma_{\theta\theta}$ (left) and $\sigma_{r\theta}$ (right) normalized with $\alpha = \frac{\sqrt{2r}}{p\sqrt{a}}$ for $n = 2$

Using extended criterion (MTS) due to Erdogan and Sih to orthotropic materials and (**??**) we conclude that:

- the inner tips start to propagate first;
- the critical values of the propagation angle $\theta_c^{(a)}$ and $\theta_c^{(na)}$, corresponding to the tips a and at na, respectively, are in a vicinity of 90^o and 270^o, respectively, and due the second hypothesis the crack tips a and na will propagate along its line .

Now, we consider the case $n = 9$ (non-interacting cracks).

Using the same procedure as in the previous cases, we plotted in Figs. 6 the normalized tangential stress $\sigma_{\theta\theta}^{(v)}$ (left) and normalized shear stress $\sigma_{r\theta}^{(v)}$ (right) corresponding to the tip $v \in \{a, na, (n+2)a\}$.

From Fig. 6 (left) we observe that all three tangential stresses have an approximative maximum value:

$$\Sigma_{\theta\theta}^{(a)} \approx \Sigma_{\theta\theta}^{(na)} \approx \Sigma_{\theta\theta}^{((n+2)a)}. \tag{4.11}$$

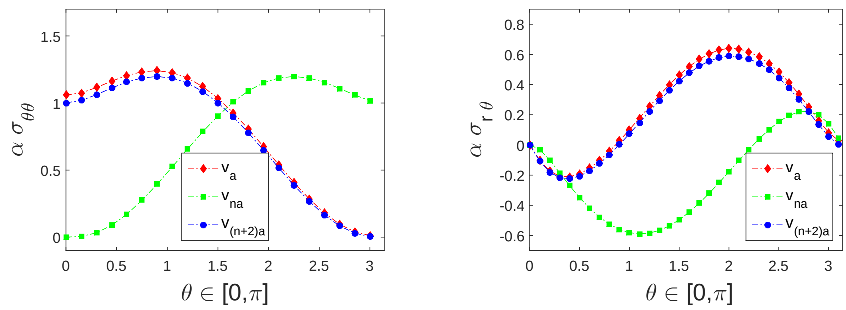

Fig. 6 $\sigma_{\theta\theta}$ (left) and $\sigma_{r\theta}$ (right) normalized with $\alpha = \frac{\sqrt{2r}}{p\sqrt{a}}$ for crack tip $n = 9$

Again, from extended (MTS) criterion and (**??**) we conclude that:

- all three tips start to propagate simultaneously along its line.

5 Conclusions

We considered an homogeneous elastic orthotropic solid containing three equal collinear cracks in mode I of fracture with uniform tensile stress.

The elastic state produced in the body is determined using the theory of Riemann-Hilbert problem by complex potentials. Analytical solution of three equal collinear cracks in an orthotropic material exerted by uniform normal stress in Mode I of fracture has been elaborated.

For a graphite-epoxy composite which contains three equal collinear cracks we extended MTS criterion due to Erdogan and Sih ([**?**], [**?**]) and we found using numerical computations which tip propagated first for two considered configurations:

- If the length of the cracks is greater than the distance between the cracks, the inner tips will start to propagate first; the propagation of crack tips is along cracks line. The cracks tend to unify and the interaction between the cracks is strong.
- If the length of the cracks is much smaller than their distance, all tips start to propagate simultaneously along cracks line. The interaction between the cracks is weak.

Appendix A

From (2.2) we get that $\Phi_j(z_j)$ can be multivalued even if $\Psi_j(z_j)$, $j=1,2$ are univalued. Consequently, to assure the uniformity of the displacement fields, we must guarantee the uniformity of the potentials $\Phi_j(z_j)$, $j=1,2$ on closed path around the two cracks.

We denote by U and V the crack tips and by Λ and Λ_j the corresponding simple closed curves around the crack (U,V) in the complex planes $z=x_1+ix_2$ and respectively $z_j=x_1+\mu_j x_2$, $j=1,2$, respectively.

According to the relations (**??**) and (**??**) the uniformity of u_1 is assured if the potentials $\Psi_j(z_j)$, $j=1,2$ satisfy the restriction:

$$\sum_{j=1}^{2}\oint_{\Lambda_j}\left(b_j\Psi_j(z_j)dz_j+\overline{b_j\Psi_j(z_j)dz_j}\right)=0. \tag{A.1}$$

Taking into account that the integrals involved in (**??**) rest unchanged if Λ and Λ_j, $j=1,2$ are changed, and squeezing the curve Λ around the crack we obtain that the restriction (**??**) is equivalent to the following one:

$$\mathrm{Re}\left\{\int_U^V\left(b_1\Psi_1^+(t)+b_2\Psi_2^+(t)\right)dt+\int_U^V\left(b_1\Psi_1^-(t)+b_2\Psi_2^-(t)\right)dt\right\}=0. \tag{A.2}$$

Using the relations (**??**) and (**??**), the uniformity of u_1 will be assured if and only if the following condition is satisfied for all three cracks:

$$\int_U^V\left(b_1\Psi_1^+(t)+b_2\Psi_2^+(t)\right)dt=\frac{\Gamma_0}{2}\left\{p\int_U^V\left(\frac{Q(t)}{i\chi(t)}-1\right)dt+\int_U^V\frac{P(t)}{i\chi(t)}dt\right\}, \tag{A.3}$$

where Γ_0 and the polynomial $Q(t)$ (see (**??**) for coefficients) are given by:

$$\Gamma_0=\frac{b_1\mu_2-b_2\mu_1}{\Delta},\quad Q(t)=\alpha_3t^3+\alpha_2t^2+\alpha_1t+\alpha_0. \tag{A.4}$$

We have (see [2,11]):

$$\operatorname{Im} \Gamma_0 = 0. \tag{A.5}$$

Since, the limit values $\Psi_1^-(t)$ and $\Psi_2^-(t)$ satisfies the eq.

$$\Psi_j^-(t) = -\Psi_j^+(t) \tag{A.6}$$

we can conclude that the condition (**??**) is fulfilled for all three cracks.

The uniformity of u_2 is assured if the condition will be satisfied

$$\sum_{j=1}^{2} \oint_{\Lambda_j} \left(c_j \Psi_j(z_j) dz_j + \overline{c_j \Psi_j(z_j) dz_j} \right) = 0. \tag{A.7}$$

The above condition is fulfilled if we have satisfied the relations

$$\int_U^V \frac{pQ(t) + P(t)}{\chi(t)} dt = 0, \tag{A.8}$$

for all three cracks, or, in the equivalent form:

$$\int_{-(n+2)a}^{-na} \frac{P(t)}{\chi(t)} dt = -p \int_{-(n+2)a}^{-na} \frac{Q(t)}{\chi(t)}, \quad \int_{-a}^{a} \frac{P(t)}{\chi(t)} dt = -p \int_{-a}^{a} \frac{Q(t)}{\chi(t)},$$

$$\int_{na}^{(n+2)a} \frac{P(t)}{\chi(t)} dt = -p \int_{na}^{(n+2)a} \frac{Q(t)}{\chi(t)} \tag{A.9}$$

where we denoted by $P(t)$ the polynomial:

$$P(t) = C_2 t^2 + C_1 t + C_0. \tag{A.10}$$

Denoting by

$$I_k = \int_{-a}^{a} \frac{t^k}{\chi(t)} dt, \quad J_k = \int_{na}^{(n+2)a} \frac{t^k}{\chi(t)} dt \tag{A.11}$$

observe that

$$\int_{-(n+2)a}^{-na} \frac{t^k}{\chi(t)} dt = (-1)^k J_k, \tag{A.12}$$

and

$$I_{2k+1} = 0, \quad k = 0, 1, 2, ... \tag{A.13}$$

with (**??**) the restrictions (**??**) take the following system of algebraic equations:

$$\begin{aligned} &J_0 C_0 - J_1 C_1 + J_2 C_2 = p(J_3 + \alpha_1 J_1) \\ &I_0 C_0 \qquad\quad + I_2 C_2 = 0 \\ &J_0 C_0 + J_1 C_1 + J_2 C_2 = -p(J_3 + \alpha_1 J_1). \end{aligned} \tag{A.14}$$

Solving the above system, we get the following values for the coefficients C_0, C_1 and C_2 of the polynomial $P(z)$:

$$C_0 = 0,\, C_1 = -p \left(\frac{J_3}{J_1} + \sigma_1 \right),\, C_2 = 0. \tag{A.15}$$

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